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Regularization of Chiral Gauge Theories

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Abstract

We propose a nonperturbative formulation of chiral gauge theories. The method involves a 'pre-regulation' of the gauge fields, which may be implemented on a lattice, followed by a computation of the chiral fermion determinant in the form of a functional integral which is regularized in the continuum. Our result for the chiral determinant is expressed in terms of the vector-like Dirac operator and hence can be realized in lattice simulations. We investigate the local and global anomalies within our regularization scheme. We also compare our result for the chiral determinant to previous exact ζ-function results. Finally, we use a symmetry property of the chiral determinant to show that the partition function for a chiral gauge theory is real.

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1 Introduction

The non-perturbative formulation of chiral gauge theories is a long-standing problem of quantum field theory with both practical and theoretical implications [1]. Recently there has been a rekindling of interest, largely due to ’tHooft [2], in the idea of using gauge field interpolation to couple lattice gauge fields to continuum fermions. By keeping the fermions in the continuum one avoids the difficulties associated with realizing chiral fermions on the lattice [3]. This idea was discussed previously in the literature in [4, 5, 6] and has now recently been further developed in [7, 8, 9].

In this approach, the gauge fields and fermions are treated differently, with the gauge fields originally defined only on the links of a spacetime lattice, as in the usual formulation of lattice gauge theory. However, an interpolation algorithm associates the gauge link variables $U_\mu(x)$ with a continuum gauge field $A_\mu(x)$ in whose background the fermion part of the functional integral is evaluated. Thus, given an interpolation scheme the problem that remains is to give a continuum formulation of the chiral determinant which can be evaluated to a desired accuracy within a finite computation.

In this paper we give a simple, gauge invariant formulation of the chiral determinant in terms of the vector-like Dirac operator which can be realized, at least in principle, in lattice simulations. The relevant background fields for our determinant have been ‘pre-regulated’ by the lattice interpolation and for the purposes of our analysis we will take them to be smooth, with variation on length scales larger than a chosen scale $\bar{a}$, where $\bar{a}$ is related to the lattice spacing a. We will discuss below to what extent various interpolation schemes satisfy this property. At the end of the procedure we allow $a, \bar{a} \to 0$, but only after first taking to infinity the continuum mode cutoff N used in the functional integral. The good behavior of the background field allows us to make well-defined manipulations of the functional integral and in particular to separate low frequency physics from the high frequency physics which comes from modes near N.

Our partition function is

$$Z \;=\; \sum_{\{U\}} e^{-S_{YM}[U]}\; det[\; \not{D}_L \;], \tag{1.1}$$

where the sum is over all gauge link configurations, the action S_{YM} is the usual Yang-Mills lattice action, and the determinant is a functional of the *continuum* gauge field $A_\mu(x)$ which is uniquely determined from each discrete set of links $\{U_\mu(x)\}$. The determinant $det[\; \not{D}_L \;]$ of the chiral Euclidean Dirac operator in the background field $A_\mu(x)$ will be defined in section 3. As we will discuss below, it is also straightforward to define regulated fermion correlators within our scheme.

Let us recall some well-known results concerning fermion determinants. In Euclidean

space fermion determinants for vector-like models are real and positive semi-definite. It is possible to convert a chiral model with fermions in representations r into a vector-like one by the addition of 'mirror' fermions which are exact copies of the originals, but in complex conjugate gauge representations r^*:

$$det[\, \not{D}_V \,] \;=\; det[\, \not{D}_L{}^r \,]\, det[\, \not{D}_L{}^{r^*} \,] \;=\; det[\, \not{D}_L{}^r \,]\, det[\, \not{D}_L{}^r \,]^* \,. \tag{1.2}$$

Thus, we see that the magnitude of the chiral determinant is simply equal to the square root of the corresponding vector-like determinant. We also see that any gauge anomaly in the chiral determinant must be a pure phase, since a vector-like model is anomaly free. In this paper we are primarily interested in models in which gauge anomalies are absent, so the important quantity is the non-anomalous phase of $det[\, \not{D}_L \,]$, which reflects the chiral nature of the model.

Our regularized chiral determinant, described in detail in section 3, is given in terms of a regularized fermion functional integral. We regulate the integral by truncating the functional measure to a finite number of modes, N. This cutoff renders the integral finite, but at the cost of introducing some gauge non-invariance in the *magnitude* of the determinant. Since, as mentioned above, we already know the desired magnitude of the fermion determinant in terms of the corresponding vector-like model, this does not lead to any ambiguity in defining the regularized model. We simply modify the magnitude to agree with the square root of the corresponding vector-like model.

We will argue that our determinant has the following properties:

(1) It correctly reproduces the known global and local anomalies, as well as allowing for anomalous phenomena such as fermion number violation. (This is in some sense 'built-in' to our formulation as it mimics the naive continuum definition as much as possible.)

(2) It yields a *gauge invariant* result in the limit $N \to \infty$, at least for infinitesimal gauge transformations.

(3) Its construction is accomplished using only the eigenvalues and eigenfunctions of the vector-like Dirac operator $\not{D}$.

This paper is organized as follows. In the following section we briefly discuss some interpolation schemes and their properties. In section 3 we give our definition of the chiral determinant in terms of a regularized fermion functional integral. In section 4 we investigate the local and global anomalies within our regularization scheme. We show that in models in which gauge anomalies cancel the phase η of our determinant is invariant under infinitesimal gauge transformations even for finite truncation to N modes. In section 5 we discuss how our results are related to previously derived exact representations of the imaginary part

of $ln(det[\ \not{D}_L\ ])$. In section 6 we discuss a simplification of the partition function which arises from the behavior of the chiral determinant under reflection of the background field. We conclude with a summary and an appendix on the convergence properties of the chiral determinant.

2 Pre-Regulation of Gauge Fields

In this section we give a brief overview of some possible interpolation schemes, with emphasis on their smoothness properties. As we will see in later sections the required size of the continuum cutoff on the fermion modes depends on the smoothness properties of the background gauge fields. It is desirable that the interpolated fields have their support in momentum space concentrated at momenta less than some scale $1/\bar{a}$, which is presumably controlled by and of order the lattice spacing a. It is also desirable that the interpolation scheme be gauge covariant, so that the effect of a lattice gauge transform on the links is consistent with effect of the interpolated gauge transform on the continuum field. More explicitly, under a gauge transform we should have

$$\begin{array}{rcl} \{U\} & \rightarrow & \{U^{\Omega}\} \\ A_\mu(x) & \rightarrow & A_\mu^{\Omega_c}(x), \end{array} \tag{2.3}$$

where Ω is the lattice gauge transform (valued only on discrete lattice sites) and Ω_c its continuum interpolation. Even given the above criteria the choice of interpolation prescription is highly arbitrary.

The interpolation scheme based on the geometrical definition of topological charge [10, 11] obeys the gauge covariance condition, but both it and the scheme based on minimizing the Euclidean action proposed in [2] produce continuum fields which are in general only piecewise continuous. Without some additional smoothing, these configurations possess Fourier transforms with support that falls off at large momentum k only like one over $|k|$ to some power. In general we would prefer configurations which are completely smooth (infinitely differentiable) and hence by the Riemann-Lebesgue lemma have Fourier transforms that fall off exponentially at large momenta.

A very simple scheme which leads to smooth continuum fields was recently given by Montvay [9]. (Unfortunately this scheme is specific to $U(1)$ gauge theories.) First one imposes a gauge fixing such as Landau gauge to define a set of gauge fields $A_{x\mu}$ on the discrete spacetime lattice. (This can always be done in such a way that the $A_{x\mu}$ are bounded by $2\pi/a$.) The Fourier coefficients $\tilde{A}_{k\mu}$ of the lattice gauge field $A_{x\mu}$ are valued only on discrete values of k_μ in the Brillouin zone: $k_\mu = 2\pi n_\mu/L_\mu$, $0 \leq n_\mu \leq L_\mu/a$. ($L_\mu$ is the extent of the lattice in the μ direction.) One can use these Fourier coefficents to define a

continuum gauge field through

$$A_\mu(x) \;=\; \frac{1}{(\prod_\nu L_\nu)} \sum_{k_\mu} e^{ik\cdot x} \;\tilde{A}_{k\mu}. \tag{2.4}$$

The resulting continuum gauge field is infinitely differentiable, and has zero support in momentum space for $|k| > 2\pi/a$. A slightly modified version of this scheme can be given [9] which also satisfies the covariance condition (2.3). An interpolation like (2.4) relates the lattice and continuum gauge transforms and guarantees that the latter also have support only in the Brillouin zone.

For our purposes in what follows, we will assume that the gauge field backgrounds in which our functional determinant is to be evaluated are bounded and infinitely differentiable, and hence that their Fourier transform falls off exponentially rapidly at momenta large compared to a scale $1/\bar{a}$ which is controlled by the lattice spacing a. We will also restrict ourselves to smooth gauge transforms $\Omega(x)$, such as would result from an interpolation of the type above from a lattice gauge transform. This implies that in the band-diagonal matrices which will appear in the next section, the entries which are outside the bands can be made zero by choice of the width of the band. In the specific case of an interpolation like that of [9], the width of the bands will be $\sim 1/a$ with entries outside the bands exactly zero.

We should note that while this assumption of smoothness of the background field allows us to discern various nice features of the determinant, such as the band-diagonal properties of certain matrices, it is not absolutely necessary to demonstrate the three main properties listed in the introduction. For that purpose the interpolations of [10, 11, 2] with their piecewise continuity should suffice.

3 Chiral Determinant

Our task is now to define the functional determinant in the smooth background $A_\mu(x)$. We will follow a straightforward procedure similar to that first used by Fujikawa [12] in his functional integral approach to the anomaly. We work in Euclidean space-time which results from the Wick rotation $x^0 \to -ix^4$ and $A_0 \to iA_4$. The Dirac operator $\not{D} \equiv \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu + A_\mu)$ becomes a Hermitian operator

$$\not{D} \;=\; \gamma^4 D_4 \;+\; \gamma^i D_i. \tag{3.1}$$

We use the convention that γ^0 is Hermitian and γ^k anti-Hermitian. The Hermitian γ^5 is defined as

$$\gamma_5 \;\equiv\; i\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; -\,\gamma^1\gamma^2\gamma^3\gamma^4. \tag{3.2}$$

The Euclidean metric is $g_{\mu\nu} \;=\; (-1,-1,-1,-1)$.

The determinant is defined, formally, through the fermion functional integral

$$det[\ \not{D}_L\ ]\ \equiv\ \int D\psi_L\ D\bar{\psi}_L\ e^{i\int d^4x\bar{\psi}_L\not{D}\psi_L}. \tag{3.3}$$

The chiral fields are given in terms of the complete set of orthonormal eigenfunctions of the Hermitian Dirac operator:

$$\not{D}\,\phi_n\ =\ \lambda_n\ \phi_n. \tag{3.4}$$

As usual, we imagine that our system has been placed in a box of size L with appropriate boundary conditions in order to discretize the eigenvalues. We define a complete set of chiral modes by

$$\begin{aligned} \phi_n^L &= \sqrt{2}P_L\ \phi_n \qquad && \lambda_n > 0 \\ &= P_L\phi_n && \lambda_n = 0, \end{aligned} \tag{3.5}$$

where $P_L\ =\ (\frac{1-\gamma_5}{2})$. The modes ϕ_n^R are defined equivalently but with left-handed projectors replaced with right-handed ones. The basis $\{\phi_n^L,\phi_n^R\}$ is complete and orthonormal, as we can see since $\not{D}\,\gamma_5\phi_n\ =\ -\lambda_n\ \gamma_5\phi_n$.

A subtle but important point should be emphasized here: the choice of positive eigenvalue modes which are used to define $\{\phi_n^L,\phi_n^R\}$ must be determined for some *fiducial* value of $A_\mu(x)$, which we will take to be $A_\mu(x)=0$. As the gauge field is varied from zero to some arbitrary configuration $A_\mu(x)$, the evolution of modes which initially had positive eigenvalues $\lambda_n \geq 0$ must be followed in order to define the new set of modes in the $A_\mu(x)$ background. In the process, some of the initially positive eigenvalue solutions may end with negative eigenvalues. It is this phenomena that is responsible for possible sign changes of the chiral determinant[†].

If we had not followed the procedure of 'tracking' the modes from the fiducial background to the background of interest, but rather applied the definition (3.5) in a naive way at each value of $A_\mu(x)$, discontinuities in functional derivatives such as $\frac{\delta}{\delta A_\mu(x)}det[\ \not{D}_L\ ]$ could arise. This sign ambiguity in the definition of the determinant is the cause of the well-known global anomaly, as first described by Witten [13] [‡]. We will return to this point in the next section when we show that our regularization of the determinant reproduces the correct results for the global anomaly.

Having chosen an orthonormal basis for the background, we can then expand

$$\begin{aligned} \psi_L(x)\ &\equiv\ \sum\ a_n\phi_n^L(x) \\ \bar{\psi}_L(x)\ &\equiv\ \sum\ \bar{b}_n\phi_n^R(x)^\dagger. \end{aligned} \tag{3.6}$$

[†]In practice, this tracking of eigenmodes would be a rather cumbersome procedure, requiring an interpolation of the background field of interest to $A_\mu(x)=0$ and the computation of the low-lying eigenmodes over this interpolation.

[‡]Fujikawa [12] is primarily interested in the effect of infinitesimal transformations on the functional measure. His formulation, which ignores the tracking of eigenmodes, is sensitive only to local anomalies.

With this expansion, the functional integral takes on a particularly simple form

$$\begin{aligned} det[\ \not{D}_L\ ] &\equiv \int \prod_n \prod_m d\bar{b}_n da_m\ e^{\ i\ \sum_n \lambda_n \bar{b}_n a_n}\ det[\ C^L\ ] \cdot det[\ C^R\ ] \\ &= det[\ C^L\ ] \cdot det[\ C^R\ ] \prod_n\ i\ \lambda_n\ . \end{aligned} \tag{3.7}$$

Fermion correlators can also be explicity evaluated, for example, the propagator

$$\begin{aligned} \langle \bar{\psi}_L(x) \psi_L(0) \rangle &= \int D\psi_L\ D\bar{\psi}_L\ (\bar{\psi}_L(x)\psi_L(0))\ e^{-\int d^4x \bar{\psi}_L \not{D} \psi_L} \\ &= \left(\sum_m \frac{1}{\lambda_m}\ \phi^R_m(x)^\dagger \phi^L_m(0) \right)\ det[\ C^L\ ] \cdot det[\ C^R\ ] \prod_n\ i\ \lambda_n. \end{aligned} \tag{3.8}$$

The factor $\prod_n \lambda_n$ is simply the square root of the corresponding vector-like functional integral (i.e. with no chiral projector in the action of (3.3)), up to the sign ambiguity we discussed previously. The factors of i in the infinite product amount to an overall phase factor which is $A_\mu(x)$ independent and are irrelevant to our analysis. We will drop them in what follows.

The additional Jacobian factors $det[\ C^L\ ]$, $det[\ C^R\ ]$ arise from the change in basis we have had to make from an initial *fiducial* basis which we define as a product over *free* modes. In other words, we choose an initial fiducial measure for the functional integral

$$\prod_n \prod_m\ d\bar{b}_n da_m, \tag{3.9}$$

where the coefficients $\bar{b}_n, a_n$ are those that arise in the expansion of (3.6) in terms of *free* solutions. In order to perform the integral using the orthonormality properties of solutions in the $A_\mu(x)$ background, we have to change basis and hence the extra Jacobian factors must appear.

The matrices C^L, C^R are defined as follows. Let the lack of an additional superscript denote free eigenmodes and the superscript A denote eigenmodes in the $A_\mu(x)$ background. Then

$$\begin{aligned} C^L_{mn} &\equiv \langle \phi^{L,A}_m | \phi^L_n \rangle \\ &= \int d^4x\ \phi^{L,A}_m(x)^\dagger \phi^L_n(x) \\ &= \int d^4x\ \phi^A_m(x)^\dagger\ P_L\ \phi_n. \end{aligned} \tag{3.10}$$

Similarly,

$$\begin{aligned} C^R_{mn} &\equiv \langle \phi^R_n | \phi^{R,A}_m \rangle \\ &= \int d^4x\ \phi^R_n(x)^\dagger \phi^{R,A}_m(x) \\ &= \int d^4x\ \phi_n(x)^\dagger\ P_R\ \phi^A_m(x). \end{aligned} \tag{3.11}$$

The C matrices are complex but unitary, so the factor $det[\ C^L\ ]\cdot det[\ C^R\ ]$ is formally a pure phase. This phase, when combined with the potential sign changes in $\prod_n\ \lambda_n$ (equivalent to phases of $i\pi$), constitute the chiral phase information mentioned in the introduction. In the absence of gauge anomalies, $det[\ \not{D}_L\ ]$ should be gauge invariant, which in turn requires that the result (3.7) is gauge invariant. At the formal level, a gauge transformation acts as a unitary transformation on the $C^{L,R}$ matrices, and hence should leave $det[\ C^L\ ]\cdot det[\ C^R\ ]$ invariant. We will examine the gauge invariance properties of our regulated version of this object in section 4.

In what follows we will regularize all of our previous expressions by eliminating all but a finite number of eigenmodes from our functional integral. This will truncate all infinite sums and products to finite ones, and also infinite matrices to finite matrices. All expressions and manipulations will then be well-defined and finite, and could in princple be implemented on a gedanken-computer. (See [7, 16] for work on similar regularization schemes.) Of course, this regularization is precisely a 'hard cutoff' in the space of eigenfunctions, and violates gauge invariance. One of our main results will be that for pre-regulated backgrounds $A_\mu(x)$, the violations of gauge invariance are limited and readily compensated. In particular, gauge non-invariance due to the finite truncation will be seen to only affect the magnitude of the chiral determinant, allowing the chiral phase information to be extracted in a well-defined manner. Our truncated expression for $det[\ \not{D}_L\ ]$ is then

$$det[\ C^L\ ]\cdot det[\ C^R\ ]\ \prod_n^N\ \lambda_n\ , \tag{3.12}$$

and for the propagator

$$\langle\bar{\psi}_L(x)\psi_L(0)\rangle\ =\ \left(\sum_m^N\ \tfrac{1}{\lambda_m}\ \phi_m^R(x)^\dagger\phi_m^L(0)\right)\ \ det[\ C^L\ ]\cdot det[\ C^R\ ]\ \prod_n^N\lambda_n. \tag{3.13}$$

where the C matrices are now finite dimensional. The number of modes that are kept is $N\ \sim\ (L\Lambda)^4$, where Λ is roughly the UV scale associated with our mode truncation. (This is up to additional factors due to degeneracies and zero modes.) Here L is the size of our box. As we will specify, the scale at which our truncation is made is determined by the smoothness scale $\bar{a}$, the lattice spacing a and the box size L.

The pre-regulation (smoothness) of the gauge field implies that the C matrices have a very simple form. We can see this by first making a useful observation about the explicit forms of the modes which appear in (3.10) and (3.11). Because the free basis is essentially a plane wave basis, it is useful to examine our eigenvalue equation (3.4) in momentum space. The eigenvalue equation for $\phi_n(q)$, has the form

$$(i\not{q}-\lambda_n)\phi_n(q)\ +\ \int d^4k\ \not{A}(q-k)\phi_n(k)\ =\ 0. \tag{3.14}$$

Here $A_\mu(k)$ represents the Fourier transform of the background field $A_\mu(x)$. Because of the smoothness property of $A_\mu(x)$, the Riemann-Lebesgue lemma tells us that $A(k) \to 0$ as $|k| \to \infty$. In fact, $A(k)$ goes to zero exponentially rapidly for $|k|\bar{a} >> 1$.

We will now show that the solution to (3.14), $\phi_n(q)$, has its support only in regions of momentum space centered around values of q which satisfy $q^2 \; = \; \lambda_n^2$. The size of those regions of support is of course determined by the properties of $A_\mu(x)$. To simplify (3.14), let us choose the Dirac basis for our gamma matrices so that γ_4 is diagonal, and the rest are off-diagonal. Now let us choose a frame in which the momentum $q_i \; = \; (0,0,0,q_4)$. We will show that if q_4 is sufficiently different from $\pm\lambda_n$, there is no solution to (3.14).

Let the the four-spinor $\phi_n(q)$ have the two-spinor components $u_n(q), v_n(q)$. In the frame we have chosen, the eigenvalue equation becomes

$$(q_4 \; + \; \lambda_n) u_n(q) \quad = \quad \int d^4k \;\; (iA_4(q-k)u_n(k) \; + \; A_i(q-k)\sigma_i v_n(k)) \tag{3.15}$$

$$(q_4 \; - \; \lambda_n) v_n(q) \quad = \quad \int d^4k \;\; (iA_4(q-k)v_n(k) \; + \; A_i(q-k)\sigma_i u_n(k))\,. \tag{3.16}$$

Multiply the top equation in (3.15) by $u_n^\dagger(q)$ and the bottom by $v_n^\dagger(q)$ and integrate both over a ball B of size $\bar{a}^{-1}$ centered about our chosen $q_i^* \; = \; (0,0,0,q_4^*)$ (so $q \; = \; q^* \; + \; q'$, and we integrate $\int_B d^4q'$ with $|q'| \; < \; \bar{a}^{-1}$).

We can then rewrite the equations as

$$\begin{aligned} (q_4^* \; - \; \lambda_n) \quad &= \quad \cdots \\ (q_4^* \; + \; \lambda_n) \quad &= \quad \cdots \; , \end{aligned} \tag{3.17}$$

where the terms on the right hand side denoted by $\cdots$ are bounded, as we will see below. This implies that there is no solution when $|q_4^* \pm \lambda_n|$ are both taken sufficiently large.

The terms on the right hand side are of three types. In matrix notation, the first type is of the form (we suppress numerical factors like π)

$$\frac{\langle u|A|u\rangle}{\langle u|u\rangle} \; < \; (L/\bar{a})^2 a^{-1}. \tag{3.18}$$

To compute the bound in (3.18) we have used Parseval's theorem,

$$\int d^4k \; |A(k)|^2 \; = \; \int d^4x \; |A(x)|^2 \; < \; L^4 a^{-2} \tag{3.19}$$

and the result that $A(k)$ has support only for momenta $|k|\bar{a} \; \lessapprox \; 1$. The second type of term is

$$\frac{\langle u|A|v\rangle}{\langle u|u\rangle} \; < \; (L/\bar{a})^2 a^{-1}, \tag{3.20}$$

where the bound applies if u_n and v_n both satisfy (3.15) and for q_4^* sufficiently different from both $\pm\lambda_n$. (If u_n and v_n were left arbitrary, one could choose $|u_n| \to 0$ while v_n remains

fixed, which would make the left hand side of (3.20) arbitrarily large. However, it is easy to see that such u_n and v_n cannot satisfy (3.15) when the lhs of those equations are sufficiently large.)

The last type of term is of the form

$$\frac{\int_B d^4q'\ q'\ u_n^\dagger(q) v_n(q)}{\int_B d^4q'\ u_n^\dagger(q) u_n(q),} \tag{3.21}$$

and is bounded by $\bar{a}^{-1}$ if again both u_n and v_n satisfy (3.15) and q_4^* is sufficiently different from both $\pm\lambda_n$.

Thus we conclude that eigenfunctions which satisfy (3.14) have support only in regions of momentum space which lie within a distance of order

$$\mathbf{max}[\ (L/\bar{a})^2 a^{-1}, \bar{a}^{-1}\ ] \tag{3.22}$$

of values q which satisfy $q^2\ =\ \lambda_n^2$.

Intuitively, we can understand this result as follows: the smooth background A has support in a compact region of momentum space, and is bounded in magnitude as well ($|A| < a^{-1}$). Its effect on eigenmodes is to 'mix-up' the original plane wave modes, but only those that are within a certain band of each other, whose size is determined by A. The overlap between modes ϕ_m, ϕ_n is zero for sufficiently large $|m-n|$. We define the number N_A such that the overlap between any modes with $|m-n|\ >\ N_A$ is zero.

The C matrices therefore have the band-diagonal form displayed below. We will always assume that the size of the matrix N is much greater than the width of the band N_A. This roughly corresponds to a choice of cutoff Λ for the regularization of our determinant. We note that from (3.22) it is clear that Λ must always be much larger than the corresponding lattice cutoff a^{-1}.

$$C^L\ ,\ C^R\ \sim\ \begin{pmatrix} \bullet & \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & 0 & 0 & 0 & 0 \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & 0 \\ 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet & \bullet \end{pmatrix} \tag{3.23}$$

For simplicity, we have adopted an index in the above matrix representing 'momentum' k rather than the index n. The latter runs from 0 to ∞ whereas the momentum can be positive or negative. We are pretending that there is only one component of momentum – in reality the C matrices are actually multidimensional with additional labels representing the

individual momenta k_i, as well as internal group indices. The multidimensional matrices have their band structure centered about vectors k_i in momentum space satisfying $\sum_{i=1}^{4} k_i^2 \ \sim\ \lambda_n^2$ for some λ_n.

Now we can see the problem that arises with truncation to a finite number of modes: the finite dimensional matrices C^L, C^R are no longer unitary. In fact, the products $C^L C^{L\dagger}$ and $C^R C^{R\dagger}$ are no longer the identity but have the following form:

$$C^L C^{L\dagger}\ ,\ C^R C^{R\dagger}\ \sim\ \begin{pmatrix} \bullet & \bullet & & & & & \\ \bullet & \bullet & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & \bullet\ \ \bullet \\ & & & & & & \bullet\ \ \bullet \end{pmatrix}, \tag{3.24}$$

Let us denote the non-diagonal submatrices in the upper left and lower right generically as $X^{L,R}$. They are of dimension N_A and are the result of a loss of unitarity that comes from our truncation. Since an infinite number of eigenfunctions are necessary to span the space of solutions, unitarity requires an infinite sum:

$$\delta_{mp}\ =\ \sum_n^{\infty} C^L_{mn} C^{L\dagger}_{np}\ =\ \sum_n^{\infty} \langle \phi_m^{L,A} | \phi_n^L \rangle\ \langle \phi_n^L | \phi_p^{L,A} \rangle. \tag{3.25}$$

From previous analysis we know that overlaps between modes m and n for which $|m-n| > N_A$ are negligible. Therefore, for $m, p < (N - N_A)$ the unitarity relation (3.25) is unaffected. On the other hand, for $(N - N_A)\ <\ m, p\ <\ N$, intermediate modes in the sum with nontrivial overlap are removed in the truncation, and hence the $X^{L,R}$ matrices in the figure above are no longer necessarily close to the identity. Note that the dimensionality of this matrix is independent of the parameter N and the mode truncation in general as long as $N >> N_A$.

From the form of (3.24), we see that the product $det[\ C^L\ ] \cdot det[\ C^{L\dagger}\ ]$ is now equal to:

$$det[\ C^L\ ] \cdot det[\ C^{L\dagger}\ ]\ =\ det[\ X^L\ ], \tag{3.26}$$

and similarly for the C^R matrices. The determinants of $X^{L,R}$ are real, as $X^{L,R}$ is Hermitian:

$$\begin{aligned}(X^L)_{mp} &= \sum_{n=N-N_A}^{N} \langle \phi_m^{L,A} | \phi_n^L \rangle\ \langle \phi_n^L | \phi_p^{L,A} \rangle \\ &= \sum_{n=N-N_A}^{N} \langle \phi_n^L | \phi_p^{L,A} \rangle\ \langle \phi_m^{L,A} | \phi_n^L \rangle \\ &= (X_L^*)_{pm}\ =\ (X^L)^\dagger_{mp},\end{aligned} \tag{3.27}$$

with similar result for X^R.

As noted previously, the formal unitarity of the infinite dimensional C matrices implies that $det[\ C^L\ ] \cdot det[\ C^R\ ]$ is a pure phase. For finite dimensional C matrices the constraint that remains from unitarity is that

$$|\ det[\ C^L\ ] \cdot det[\ C^R\ ]\ |^2 \ =\ det[\ X^L\ ] \cdot det[\ X^R\ ] \ \equiv\ det[\ X\ ]^2, \tag{3.28}$$

which allows for an arbitrary phase in $det[\ \not{D}_L\ ]$, but does not restrict the magnitude to be unity. Here we define X as a diagonal matrix $\mathrm{diag}\{\lambda_1^X, \cdots, \lambda_{N_A}^X\}$, where the λ_n^X are given by the square root of the product of the corresponding n-th eigenvalues of X^L, X^R. X is Hermitian and $det[\ X\ ]$ is real, so we can still extract the phase unambiguously from the finite dimensional C matrices, where

$$det[\ C^L\ ] \cdot det[\ C^R\ ] \ \equiv\ e^{i\eta}\ det[\ X\ ] \tag{3.29}$$

or

$$\eta \ \equiv\ \mathrm{Im}\left(ln\left(det[\ C^L\ ] \cdot det[\ C^R\ ]\right)\right) \quad . \tag{3.30}$$

In the next section we will examine the behavior of $det[\ C^L\ ] \cdot det[\ C^R\ ]$ under gauge transformations. Any anomaly, being a pure phase, resides in the factor $e^{i\eta}$. In the absence of gauge anomalies, η is gauge invariant, which is equivalent to the requirement that under a gauge transformation, the change in $det[\ C^L\ ] \cdot det[\ C^R\ ]$ is purely real. We will see explicitly that this is the case in section 4.

Our final result for the regularized chiral determinant is

$$det[\ \not{D}_L\ ]_{reg} \ =\ e^{i\eta}\ \prod_n^N\ \lambda_n\ , \tag{3.31}$$

where the phase η is gauge invariant in the absence of gauge anomalies and must be extracted from the finite dimensional C matrices via (3.29) or (3.30). In the appendix we discuss the convergence properties of η. It appears that η is determined by the $N_A \times N_A$ submatrices of $C^{L,R}$ (i.e. it depends on the properties of eigenmodes with $n\ \lesssim\ N_A$), and therefore converges to a well-defined value as $N \to \infty$. An alternate method of extracting the complex phase in (3.31) already exists, as we will discuss in section 5.

4 Local and Global Anomalies

In this section we examine the local and global anomalies within our regularization scheme. We will verify that our regularization scheme leads to the usual anomalous Ward-Takahashi (WT) identities in the limit that $N \to \infty$ and that in the absence of anomalies the phase factor η is gauge invariant. Finally, we shall examine how Witten's global anomaly is manifested in our scheme.

First let us review the behavior of the functional measure under rotations of the fermion fields [12]:

$$\begin{aligned} \psi_L &\rightarrow e^{-i\alpha(x)}\psi_L \\ \bar{\psi}_L &\rightarrow \bar{\psi}_L e^{i\alpha(x)}. \end{aligned} \tag{4.1}$$

These rotations can also correspond to non-Abelian gauge transformations if we allow $\alpha(x)$ to be an element of the Lie group: $\alpha(x) = \alpha^a(x)T^a$. Any subtle effects from such a transformation are to be found in the functional measure, or equivalently in the Jacobian determinants $det[\ C^L\ ] \cdot det[\ C^R\ ]$. As usual, for infinitesimal rotations we can write

$$\begin{aligned} det[\ C^L\ ] &= det[\ \delta_{mn} + i\int d^4x\ \alpha(x)\phi_m^L(x)^\dagger\phi_n^L(x)] \\ &= exp[\ i\sum_n\int\ d^4x\ \alpha(x)\phi_n^L(x)^\dagger\phi_n^L(x)] \end{aligned} \tag{4.2}$$

and similarly,

$$det[\ C^R\ ]\ =\ exp[-i\sum_n\int d^4x\ \alpha(x)\phi_n^R(x)^\dagger\phi_n^R(x)]. \tag{4.3}$$

Combining these equations gives

$$\begin{aligned} det[\ C^L\ ]\ det[\ C^R\ ] &= exp[\ i\sum_n\int d^4x\ \alpha(x)(\phi_n^L(x)^\dagger\phi_n^L(x)\ -\ \phi_n^R(x)^\dagger\phi_n^R(x))\ ] \\ &= exp[\ i\sum_n\int d^4x\ \alpha(x)\phi_n(x)^\dagger\gamma_5\phi_n(x)]. \end{aligned} \tag{4.4}$$

In Fujikawa's scheme [12], the infinite sums are regulated by the insertion of a convergence factor of

$$f(\not{D}^2/\Lambda^2)\ =\ f(\lambda_n^2/\Lambda^2), \tag{4.5}$$

where $f(0) = 1$ and $f(\infty) = f'(\infty) = f''(\infty)\ \cdots\ = 0$. The result is unchanged as long as the function f is smooth and obeys the above boundary conditions. The usual choice for $f(\not{D}^2/\Lambda^2)$ is $f(\not{D}^2/\Lambda^2) = exp(-\not{D}^2/\Lambda^2)$.

Using Fujikawa's result in the limit $N \rightarrow \infty$ (which corresponds to taking $\Lambda \rightarrow \infty$) yields the well-known result

$$\lim_{\Lambda\rightarrow\infty}\ \sum_n^\infty \phi_n(x)^\dagger\gamma_5\phi_n(x)\ f(\lambda_n^2/\Lambda^2)\ =\ -\tfrac{1}{16\pi^2}\ F\tilde{F}(x). \tag{4.6}$$

If the original transformation (4.1) had been a non-Abelian gauge transformation, (4.6) would have additional color structure and be proportional to $Tr[\{T^a, T^b\}T^c]$.

One could instead have taken the function $f(\not{D}^2/\Lambda^2)$ to approach a step-function, so as to reproduce a hard mode cutofff [16, 7]. Since the result is independent of the detailed form of f, we obtain the usual anomalous WT identity for the regulated current

$$\partial_\mu J_\mu\ =\ \tfrac{1}{16\pi^2}\ F\tilde{F}(x). \tag{4.7}$$

Note that the fermionic currents $J_\mu(x)$ are themselves divergent objects (they involve a product of operators evaluated at the same spacetime point x) which require regularization and some choice of subtraction in their definition. In our scheme any correlator is regulated automatically and non-locally by the mode truncation. For example, see (3.13). As $x \to 0$ the usual short-distance divergence is cut-off by the truncation of the series at N.

From (4.4) we see that the absence of anomalies implies that the phase factor η defined by the $N \to \infty$ limit of (3.30) is gauge invariant, at least under infinitesimal rotations. The corrections that would appear on the RHS of (4.6) due to a finite truncation are of the form, e.g., TrF^k/M^{2k-4} ($k > 2$) and vanish as N is taken to infinity. What is more, the corrections correspond to local operators and can be compensated by proper choice of local counterterms if desired.

Let us now consider the global anomaly [13], which arises in $SU(2)$ gauge theories due to the fact that the fourth homotopy group of $SU(2)$ is nontrivial,

$$\pi^4(SU(2)) \;=\; Z_2. \tag{4.8}$$

This means that in four dimensional Euclidean space there is a gauge transformation $\Omega(x)$ such that $\Omega(x) \to 1$ as $|x| \to \infty$, and $\Omega(x)$ covers the gauge group in such a way that it cannot be continuously deformed to the identity. The so-called mod two Atiyah-Singer index theorem [14] then implies that as a gauge background $A_\mu(x)$ is changed continuously to its gauge transform $A^\Omega_\mu(x)$ (e.g. via $A^t_\mu(x) \;=\; A^\Omega_\mu(x) \,+\, (1-t)\,(A_\mu(x) \,-\, A^\Omega_\mu(x))$ as t goes from zero to one), an *odd* number of positive-negative pairs of eigenvalues of $\not{D}$ will switch places. (The set of eigenvalues must match exactly at $t = 0$ and $t = 1$ since $A_\mu(x)$ and $A^\Omega_\mu(x)$ are related by a gauge transform.) This leads to an overall change in sign in the product

$$\prod_n^\infty \lambda_n \tag{4.9}$$

as long as we choose our eigenvalues in the way we have described in the previous section, tracking modes continuously beginning from some fiducial background $A_\mu(x)$.

Now, given that we are only interested in backgrounds which arise from our pre-regulation procedure, it is easy to see that any modes which switch places are within roughly N_A of $n = 0$. Therefore, as long as $N >> N_A$, the truncated product

$$\prod_n^N \lambda_n \tag{4.10}$$

which appears in our regulated determinant undergoes the same sign change as the infinite product above when the background field is transformed by $\Omega(x)$. (Of course, this $\Omega(x)$ must relate two backgrounds A and A' which satisfy our smoothness conditions, so $\Omega(x)$ itself must be smooth.) The modes near the cutoff, $N - N_A \;<\; n \;<\; N$, are not necessary to reproduce this result.

5 Exact Representations

In this section we compare our form of the determinant to exact representations previously obtained using ζ-function methods [5, 15]. We will not give the derivation of the results of [5, 15] here, since the details are somewhat technical, but will merely state them. Suppose we wish to compute the imaginary part of the chiral effective action $ln(det[\ \not{D}_L\ ])$ in the background $A_\mu(x)$ relative to $ln(det[\ \not{D}_L\ ])$ in the fiducial background $A_\mu(x) = 0$ (we take the latter to have no phase as a choice of convention). First form the five dimensional background gauge field A_t which interpolates adiabatically between $A_\mu(x) = 0$ and our chosen $A_\mu(x)$ as the parameter t varies from $-\infty$ to $+\infty$. Next, consider the 5-dimensional Dirac operator defined by

$$\not{D}_5 \ = \ (\ i\gamma^5 \partial_5 \ + \ \not{D}_4\ ), \tag{5.1}$$

where $\not{D}_4$ is the vector-like Dirac operator in four dimensions. The result in an anomaly free model is the following:

$$\mathrm{Im}\ ln(det[\ \not{D}_L\ ]) \ = \ \pi\ (\ \eta(0)\ +\ \mathrm{dim\ ker}\ \not{D}_5\ ), \tag{5.2}$$

where $\eta(0)$ is the famous 'η-invariant' of the Atiyah-Singer Index theorem. It is given by the analytic continuation to $s = 0$ of

$$\eta(s) \ = \ \sum_{\lambda \neq 0} \tfrac{\mathrm{sign}(\lambda)}{|\lambda|^{-s}}, \tag{5.3}$$

where λ denotes eigenvalues of $\not{D}_5$. One can think of $\eta(0)$ as the regularized spectral asymmetry of $\not{D}_5$:

$$\eta(0) \ \sim \ \sum_{\lambda > 0} 1 \ - \ \sum_{\lambda < 0} 1. \tag{5.4}$$

The term dim ker $\not{D}_5$ simply counts the number of zero modes of the operator $\not{D}_5$. We see that the determinant changes sign whenever such a zero mode appears. An identical change in sign can be seen to result in our treatment if we recall Witten's result [13], using a construction like the five dimensional one above, that a zero mode of $\not{D}_5$ implies that in the interpolated background A_t the flow of eigenvalues of $\not{D}_4$ is such that an *odd* number of eigenvalue pairs change sign. Thus the sign change in due to dim ker $\not{D}_5$ in the exact representation matches that due to eigenvalue flow in our form of the determinant.

This leaves the phase η defined in (3.30) to be identified with $\eta(0)$. This identification is extremely nontrivial mathematically, as it relates the η-invariant in five dimensions to some rather detailed properties of the eigenfunctions of the four dimensional Dirac operator. As mentioned previously, we have not rigorously proved (although it is plausible - see the appendix) that our phase η converges to a well-defined limit as the cutoff is taken to infinity.

On the other hand, $\eta(0)$ is defined in terms of an analytic continuation which, while mathematically sound, may or may not have the same physical content as our η [§]. These issues clearly deserve further investigation.

6 Reflections and Phases

In this section we make an observation which simplifies the form of the partition function (1.1). We show that despite the possibility of complex phases in the chiral determinant, in the absence of gauge anomalies the partition function itself is real. Our main observation is that the $\eta(0)$ part of the fermion effective action changes sign when the gauge background is reflected through any of the hyperplanes: $\{x_\mu \ = \ 0\}$ – in other words, it has odd parity. This implies that all imaginary parts eventually cancel in (1.1), leading to a real partition function.

Consider a background $A_\mu(x)$ and its reflection through $x_4 = 0$:

$$A^*_\mu(x_i, x_4) \ \equiv \ A_\mu(x_i, -x_4). \tag{6.1}$$

The five dimensional interpolations ${}^tA_\mu(x)$ and ${}^tA^*_\mu(x)$ will also be reflections of each other. We will specialize to the temporal gauge: $A_4 = 0$. We do not lose any generality by doing so if we are working in a model without gauge anomalies, since in that case $det[\ \not{D}_L\ ]$ is gauge invariant. Note that the pure gauge actions $S_{YM}[A]$ and $S_{YM}[A^*]$ are identical.

We want to show two things:

(I) The phases of the determinant $det[\ \not{D}_L\ ]$ in the backgrounds $A_\mu(x)$ and $A^*_\mu(x)$ are related by a minus sign. In other words,

$$\eta(0)|_{\ {}^tA} \ = \ - \ \eta(0)|_{\ {}^tA^*} \ \ , \tag{6.2}$$

which is equivalent to showing that there is a mapping of $\lambda_n \ \to \ -\lambda_n$ when the gauge background is reflected through $x_4 \ = \ 0$.

(II) The eigenvalues of $\not{D}_4$ are invariant under $A \ \to \ A^*$. This should be clear since the Euclidean Dirac operator can be regarded as a Hamiltonian, and the energy eigenvalues are invariant under reflection of the gauge background. We will also see this explicitly below by looking at eigenfunctions and their eigenvalues.

[§] In the work of Ball and Osborn [15] similar results are derived using Pauli-Villars rather than ζ-function regularization.

Given points (I) and (II), we can arrange the partition function in the following way:

$$\begin{aligned} Z &= \sum_{\{A\ ,\ A^*\}} e^{-S_{YM}[A]}\ det[\ \not{D}_L\ ] \\ &= \sum_{\{A\}} e^{-S_{YM}[A]}\ 2\ Re(\ det[\ \not{D}_L\ ]\ ) \\ &= \sum_{\{A\ ,\ A^*\}} e^{-S_{YM}[A]}\ Re(\ det[\ \not{D}_L\ ]\ ), \end{aligned} \tag{6.3}$$

where first and third sums are over all configurations and the second sum is only over the half of the possible configurations which remain after modding out by the Z_2 reflection symmetry. We have also used the fact that the sign of $det[\ \not{D}_L\ ]$ is the same in the A and A^* backgrounds. This follows from (II) and the tracking of eigenvalues.

Now to the proof of (I). The eigenvalue equation $\not{D}_5\phi_n\ =\ \lambda_n\phi_n$ is as follows (we use the chiral basis for gamma matrices):

$$[+\ i\partial_5 u_n\ -\ i\partial_4 v_n\ +\ D_i\sigma_i v_n]\ =\ \lambda_n u_n \tag{6.4}$$

$$[-\ i\partial_5 v_n\ -\ i\partial_4 u_n\ -\ D_i\sigma_i u_n]\ =\ \lambda_n v_n. \tag{6.5}$$

One can check that, corresponding to the original eigenfunction of $\not{D}_5$ in the ${}^tA$ background,

$$\begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix}, \tag{6.6}$$

is an eigenfunction of $\not{D}_5$ in the background ${}^tA^*$, with eigenvalue $-\lambda_n$:

$$\begin{pmatrix} v_n(x_i, -x_4, x_5) \\ u_n(x_i, -x_4, x_5) \end{pmatrix}. \tag{6.7}$$

This shows that the eigenvalue spectrum of $\not{D}_5$ in the reflected background is the negative of the original spectrum, which is sufficient to prove (I).

We can see that this result also follows from the functional integral form of the determinant if we continue to work in the temporal gauge. In this gauge a parity-like operation relates the bases $\{\phi_n^L(x)\}$ and $\{\phi_n^R(x)\}$ in backgrounds which are related by a reflection through one of the hyperplanes $\{x_\mu\ =\ 0\}$.

Given an eigenfunction (in the chiral basis where $\gamma_5\ =\ \mathrm{diag}\{1,-1\}$) of the Dirac operator in the background A:

$$\phi_n(x)\ =\ \begin{pmatrix} \phi_n^R(x) \\ \phi_n^L(x) \end{pmatrix}, \tag{6.8}$$

the following spinor function is an eigenfunction with the same eigenvalue, but of the Dirac operator in the background A^*:

$$\phi'_n(x)\ =\ \begin{pmatrix} -\phi_n^L(\vec{x}, -x_4) \\ \phi_n^R(\vec{x}, -x_4) \end{pmatrix}. \tag{6.9}$$

(Note that this proves point (II) above.) This similarity between the left and right chiral bases in the two backgrounds yields the following matrix relations:

$$\begin{aligned} C^L(A) &= \{C^R(A^*)\}^* \\ C^R(A) &= \{C^L(A^*)\}^* \; . \end{aligned} \tag{6.10}$$

Together, these imply that

$$det[\ C^L(A)\ ] \cdot det[\ C^R(A)\ ] \ = \ \{det[\ C^L(A^*)\ ] \cdot det[\ C^R(A^*)\ ]\}^*, \tag{6.11}$$

which is equivalent to (I).

We should note a limitation of the result: the insertion of an operator into the sum in (6.3) (e.g. to compute an n-point correlator) will in general destroy the reflection symmetry of the terms in the sum. Therefore we can only compute correlators using the weighting $Re(\ det[\ \not{D}_L\ ]\ )$ if the operators are themselves invariant under at least one reflection, such as

$$\langle\ O(x_i, x_4)\ +\ O(x_i, -x_4)\ \rangle, \tag{6.12}$$

where O is a generic field operator. This does not seem to be a significant limitation on our ability to investigate issues of interest such as chiral symmetry breaking, confinement or particle spectra.

7 Conclusions

We have proposed a formulation of chiral gauge theory which should allow, in principle, the evaluation of any quantity to a specified accuracy within a finite computation. In order to avoid the problems associated with lattice chiral fermions [3], we employ an interpolation of the original lattice gauge fields to the continuum [2, 4, 5]. All of the fermionic aspects of the theory have been expressed here in terms of the eigenvalues and eigenfunctions of the vector-like Dirac operator, which can itself be implemented directly on a lattice if desired.

The main focus of our investigation was the chiral determinant defined in terms of a regulated continuum functional integral. When the gauge field background is suitably well-behaved (e.g. resulting from lattice interpolations like those described in section 2), we find that an object with all the desired properties can be extracted from the functional integral as long as the mode cutoff N used to regulate the integral is kept sufficiently large. For well-behaved background gauge fields, any violations of gauge invariance introduced by a hard mode cutoff are confined to modes close to the cutoff. These violations of gauge invariance do not affect the complex phase information which characterizes the chiral nature of the theory. The order of limits that are necessary for our regulator are as follows: first, the continuum cutoff $N \to \infty$, followed by the original gauge lattice spacing $a \to 0$.

Our result for the chiral determinant is (from (3.31))

$$det[\ \not{D}_L\ ]_{reg}\ =\ e^{i\eta}\ \prod_n\ \lambda_n\ , \tag{7.1}$$

where the phase η is defined in (3.30) and the eigenvalues λ_n are described below (3.5). Both η (in the absence of anomalies and in the limit $N \to \infty$) and the eigenvalues are gauge invariant functions of the background field. In the appendix we give some evidence that suggests that $det[\ \not{D}_L\ ]$ as defined above will converge to a well-defined value as $N \to \infty$, with its value mainly dependent on modes with $n \lessapprox N_A$. (7.1) is consistent with the exact representation derived using ζ-function regularization, given the identification of the phases η and $\eta(0)$ described in section 5. One difference between the functional integral formulation and the ζ-function regularization is that in the former the physical aspects of the chiral determinant such as level crossing and the role of high versus low frequency modes are more transparent.

Using our result for the chiral determinant we showed that despite the sum over complex phases, the partition function for a chiral gauge theory is real. The simplified result for the partition function that we derived in section 6 still requires the computation of the non-anomalous phase η, as

$$Re(\ det[\ \not{D}_L\ ]_{reg}\ )\ =\ cos(\eta)\ \prod_n \lambda_n\ . \tag{7.2}$$

Therefore its main advantage is that real rather than complex terms may be summed to yield the final result. It still remains a formidable technical problem (although not one of principle) to compute the phase η on the lattice.

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8 Appendix

In this appendix we discuss the behavior of Dirac eigenfunctions for large λ_n and implications for the convergence properties of the chiral determinant.

We begin by considering the two component Dirac eigenvalue equation (3.15), recopied here for convenience.

$$\begin{aligned}(q_4 \; + \; \lambda_n) u_n(q) \quad &= \quad \int d^4k \; \left(iA_4(q-k) u_n(k) \; + \; A_i(q-k)\sigma_i v_n(k) \right) \\ (q_4 \; - \; \lambda_n) v_n(q) \quad &= \quad \int d^4k \; \left(iA_4(q-k) v_n(k) \; + \; A_i(q-k)\sigma_i u_n(k) \right) . \end{aligned} \tag{8.1}$$

Consider the limit that λ_n becomes arbitrarily large (in particular, we want $n >> N_A$). We can see immediately that either $v_n(q)$ or $u_n(q)$ must approach zero in this limit, depending on the sign of q_4. Let us choose $q_4 < 0$ so that $u_n(q)$ is nonzero. The equations in (8.1) then reduce to

$$(q_4 \; + \; \lambda_n) u_n(q) \; = \; \int d^4k \; iA_4(q-k) u_n(k) \tag{8.2}$$

up to corrections which vanish as $\lambda_n \to \infty$. Note that we can set A_4 to zero by a suitable gauge transform (which may be different for different solutions due to the special Lorentz frame we have chosen above). Thus the eigenfunctions in this limit reduce to free solutions, up to a gauge transform. We therefore have

$$\lim_{m,n \to \infty} \; C^L_{mn} \; = \; (C^R_{mn})^* \; , \tag{8.3}$$

for arbitrary backgrounds $A_\mu(x)$ which are not necessarily pure gauge. Since the eigenvalues of (8.1) are gauge invariant, we also learn that

$$\lim_{n \to \infty} \; \lambda^A_n \; = \; \lambda^0_n . \tag{8.4}$$

There are several consequences of the results (8.3) and (8.4). Consider the Jacobian factor

$$det[\; C^L \;] \cdot det[\; C^R \;] \; = \; det[\; C^L (C^R)^T \;] . \tag{8.5}$$

(8.3) implies that for $N_A < \; m,n \; < N - N_A$ the matrix $\{ C^L (C^R)^T \}_{mn}$ is close to the identity δ_{mn}, and hence does not contribute to the phase η. From (3.27) we know that the diagonal corners of the matrix $N - N_A < \; m,n \; < N$ are Hermitian. Thus we expect that the non-anomalous phase η should roughly depend only on the $N_A \times N_A$ submatrices of $C^{L,R}$ and that η converges to a well-defined limit as $N \to \infty$ for fixed a. This is intuitively plausible, since it is mainly modes with $n \lesssim N_A$ which are affected by the presence of the pre-regulated background field.

The result (8.4) is relevant to the existence and convergence of the limit

$$\lim_{N \to \infty} \; \prod_n^N \; \frac{\lambda^A_n}{\lambda^{A'}_n} \tag{8.6}$$

where A and A' are different background fields. The existence of (8.6) requires that very large eigenvalues are essentially unperturbed by a sufficiently well-behaved background $A_\mu(x)$ or $A_\mu(x)'$. Without this property, the weighting factor of gauge configurations in (1.1) would be extremely difficult to compute and could exhibit drastic oscillations due to small changes in background field $A_\mu(x)$. This question is actually also relevant to the the fermion determinant in vector-like models like QCD, and is not specific to chiral models. Unquenched lattice QCD computations assume that the ratio (8.6) converges to its limit for eiqenvalues of order the lattice spacing.

Closer examination of (8.1) reveals that large eigenvalues behave as

$$\lambda_n^A \ \sim \ \lambda_n^0 \ + \ \mathcal{O}(A^2/\lambda_n^0). \tag{8.7}$$

Using $\lambda_n^0 \sim (n^{1/4}/L)$, where L is the size of our box, we have that

$$ln\left(\prod_n^N \frac{\lambda_n^A}{\lambda_n^{A'}}\right) \ \sim \ \sum_n^N \mathcal{O}\left(\frac{1}{n^{1/2}}\right). \tag{8.8}$$

It is plausible that oscillations in the signs of terms in the sum (8.8) allow it to converge. This would imply that (8.6) is well-defined.

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